Variational Principles for Non-symmetric Markov Chains

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CONTENTS

Non-symmetric Markov chains

- Irreducible discrete time Markov chain $X = (X_n)_{n\in\mathbb{N}}$ on countable state space S with transition matrix P.
- $\alpha > 0$ is an excessive measure: $\alpha_i \geq \sum_{j\in S} \alpha_j p_{ji}, i \in S$. Define $\widehat{P}$

$$\widehat{p}_{ij} := \frac{\alpha_j p_{ji}}{\alpha_i},\ i, j \in S.$$

  P is symmetric with respect to α, if

$$P = \widehat{P}.$$

- Non-symmetric Markov chains are difficult to deal with than the symmetric ones.

Existing Results

Doyle(1994)[1], Gaudillière and Landim(2014)[2] obtained the variational principle of the capacity between two disjoint sets.
Huang and Mao(2018)[3] gave the variational principle of hitting time for ergodic Markov chains.
Huang and Mao(2019+)[4] got the variational formulas of asymptotic variance.

[1] Doyle P.G.. Energy for Markov chains. http://www.math.dartmouth.edu/doyle, 1994

[2] Gaudillière A., Landim C.. A Dirichlet principle for non reversible Markov chains and some recurrence theorems. Probability Theory and Related Fields, 2014, 158: 55-89

[3] L.-J. Huang, Y.-H. Mao. Variational Principles of Hitting times for Non-reversible Markov Chains. Journal of Mathematical Analysis and Applications, 2018, 468-2:959-975

[4] L.-J. Huang, Y.-H. Mao.Variational Formulas for Asymptotic Variance of Markov Chains. Preprint

Dual Poisson Equation

- Poisson equation
  For any non-trivial subset A of S, $\xi, \widehat{\xi} \geq 0$, we consider poisson equation:

$$\begin{cases} (I-P)x(i) = \xi(i), & i \in A^c; \\ x(i) = \eta(i), & i \in A. \end{cases} \tag{1}$$

- Expression of solution
  Define $\tau_A = \inf\{n \geq 0 : X_n \in A\}$. When

$$\varphi(i) := \mathbb{E}_i \left[\sum_{n=0}^{\tau_A - 1} \xi(X_n) + \eta(X_{\tau_A}) \right]$$

  is well defined, it is a solution of equation (1).
- Dual Poisson equation

$$\begin{cases} (I-\widehat{P})x(i) = \widehat{\xi}(i), & i \in A^c; \\ x(i) = \widehat{\eta}(i), & i \in A. \end{cases} \tag{2}$$

Variational Principle of Hitting Times

- Huang and Mao(2018)[1] obtained the variational principle of hitting time for ergodic Markov chains with stationary distribution π:

$$\frac{1}{\mathbb{E}_\pi[\tau_A]} = \inf_{f|_A=0,\pi(f)=1} \sup_{g|_A=0,\pi(g)=0} \langle f-g, (I-P)(f+g)\rangle_\alpha,$$

where $\pi(f) := \sum_{i\in S} \pi_i f_i$ and $\langle f, g\rangle_\alpha := \sum_{i\in S} \alpha_i f_i g_i$.

- Aim:

$$\tau_A(\xi\equiv 1, \eta\equiv 0) \to \sum_{n=0}^{\tau_A-1} \xi(X_n)(\eta\equiv 0)$$

[1] L.-J. Huang, Y.-H. Mao. Variational Principles of Hitting times for Non-reversible Markov Chains. Journal of Mathematical Analysis and Applications, 2018, 468-2:959-975

Notations

$\mathcal{K}$ denotes the space of functions with finite support

$$\mathscr{S}(f) = \{i : f(i) \neq 0\}.$$

And let $L^2(\alpha)$ be the space of square summable functions endowed with the scalar product:$\langle f, g\rangle_\alpha$. And define $\langle f, g\rangle := \sum_{i\in S} f_i g_i$.
Let ν be an initial distribution such that there is a solution, denoted by $\widehat{\varphi}$, of the following equation:

$$\begin{cases} (I - \widehat{P})x(i) = \frac{\nu_i}{\alpha_i}, & i \in A^c; \\ x(i) = 0, & i \in A. \end{cases} \tag{3}$$

Variational Principle of Addictive Functional

Theorem

If $\widehat{\varphi}$ *and* $\varphi := (\mathbb{E}_i \sum_{n=0}^{\tau_A - 1} \xi(X_n))_{i \in S}$ *belong to* $L^2(\alpha)$, *then*

$$\frac{1}{\mathbb{E}_{\nu} \sum_{n=0}^{\tau_A - 1} \xi(X_n)} = \frac{1}{\langle \widehat{\varphi}, (I - P)\varphi \rangle_{\alpha}} = \inf_{f \in \mathcal{F}_A} \sup_{g \in \mathcal{G}_A} \langle f, (I - P)g \rangle,$$

where $\mathcal{F}_A = \{f \in \mathcal{K} : f|_A = 0, \sum_{i \in A^c} f_i \xi_i = 1\}$,

$$\mathcal{G}_A = \{g \in \mathcal{K} : g|_A = 0, \sum_{i \in A^c} \nu_i g_i = 1\}.$$

Definition of Capacity for Transient Markov Chains

For transient P, define $N := \sum_{n=0}^{\infty} P^n$.

- Equilibrium set
  For any subset E of S, let $\tau_E^+ := \inf\{n \geq 1 : X_n \in E\}$ be first return time of E.
  Define the escape function $e_i := \mathbb{P}_i(\tau_E^+ = \infty)\mathbf{1}_E(i)$.
  A set E is an equilibrium set if $\sum_{i \in E} \alpha_i e_i < \infty$ and for any initial distribution, the set E is entered only finitely often a.s.
- Capacity of a equilibrium set E

$$C(E) := \sum_{i \in E} \alpha_i e_i$$

Variational Principle of Capacity for Symmetric Markov Chains

- In Kemeny, Snell and Knapp(1976)[1], there is a variational principle of capacity for symmetric transient Markov Chains: for any equilibrium set E,

$$C(E) = I_t(e) = \inf_{f \in \mathscr{F}_E} I_t(f),$$

where

$$I_t(f) := \langle f, Nf \rangle_\alpha,$$

$$\mathscr{F}_E := \{f : f_{E^c} = 0, \alpha(f) = C(E), \langle f, Nf \rangle_\alpha < \infty\}.$$

- Aim:

symmetric $\rightarrow$ non-symmetric

[1] J.G.Kemeny, J.L.Snell and A.W.Knapp. Denumerable Markov chains, 2nd.ed. Springer-Verlag, New York, 1976.

Variational Principle of Capacity for Non-symmetric Markov chains

For any equilibrium set E and $f \in \mathcal{F}_E$, denote $\mathcal{F}_f := \{g \in \mathcal{F}_E : \langle g, Nf \rangle_\alpha < \infty\}$. We generalize the above result to non-symmetric Markov chains.

Theorem

Assume P is transient, then for any equilibrium set E,

$$C(E) = \inf_{f \in \mathscr{F}_E} \sup_{g \in \mathscr{F}_f} \langle g, Nf \rangle_\alpha.$$

For recurrent P, we can also get the variational principle.

Connection to Poisson Equations

Define the hitting function h:

$$h_i := \mathbb{P}_i(\tau_E < \infty).$$

The poisson equation related to the capacity is:

$$(I - P)h = e.$$

Future Works

- Life time
  E.Nummelin(1991)[1]: Let L denote the life time of X
  $$L = \sup\{n \geq 0 : X_n \in S\}.$$
  Suppose that $\mathbb{P}_{i_0}\{L < \infty\} = 1$ for some $i_0 \in S$. Let φ be a bounded solution of
  $$(I - P)\varphi = \xi.$$
  Then $\varphi = \mathbb{E}_\bullet \sum_0^L \xi(X_n)$ on $\{i \in S : P_i\{L < \infty\} = 1\}$.
- Compare different excessive measures

[1] E.Nummelin.On the Poisson Equation in the Potential Theory of a Single Kernel. Mathematica Scandinavica, 1991, 68:59-82

Characterization of excessive measures for transient chains

Define

$$e_{ji}^{(n)} = \mathbb{P}_j[X_n = i, X_m \neq j, 0 < m < n], \qquad e_{ji} = \sum_{n=1}^{\infty} e_{ji}^{(n)}.$$

$$\widehat{e}_{ji} = \begin{cases} e_{ji}, & i \neq j : \\ 1, & i = j. \end{cases}$$

Theorem

$\mathcal{E} :=$ {*excessive measures of* P},
$\mathcal{E}_1 :=$ {*Non-negative linear combination of* $(\widehat{e}_{ji})_{i \in E}$},
$\mathcal{E}_2 := \{\nu N | \nu$ *is a measure*$\}$, *then*

$$\mathcal{E} = \mathcal{E}_1 = \mathcal{E}_2.$$

THANKS !